

ISOLATED SUBGROUPS OF FINITE ABELIAN GROUPS

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Abstract. We say that a subgroup H is isolated in a group G if for every $x \in G$ we have either $x \in H$ or $\langle x \rangle \cap H = 1$. We describe the set of isolated subgroups of a finite abelian group. The technique used is based on an interesting connection between isolated subgroups and the function sum of element orders of a finite group.

Keywords: finite abelian group; isolated subgroup; sum of element orders

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1. INTRODUCTION

Let G be a finite group. We say that a subgroup H of G is isolated in G if for every $x \in G$ we have either $x \in H$ or $\langle x \rangle \cap H = 1$. Groups with isolated subgroups were studied in [2], [3]. However, this concept appears much earlier (see for instance Section 66 of [7] and the entry “isolated subgroup” in Encyclopedia of Mathematics, cf. [5]). The starting point for our discussion is given by Janko’s paper (see [6]) that investigates isolated subgroups for certain classes of nonabelian p -groups.

In the following, we determine these subgroups for finite abelian groups. The problem is reduced to finite abelian p -groups. Our main result can be summarized as follows.

Theorem 1.1. *Let p be a prime number and $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p -group, where $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$.*

- (a) *If $\alpha_1 > 1$, then the unique isolated subgroups of G are 1 and G .*
- (b) *If $1 = \alpha_1 = \alpha_2 = \dots = \alpha_r < \alpha_{r+1} \leq \dots \leq \alpha_k$ and $A = \mathbb{Z}_{p^{\alpha_{r+1}}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$, then the isolated subgroups of G are G and all subgroups $H \leq G$ with $H \cap A = 1$.*

The main tool used is the function sum of element orders of G ,

$$\psi(G) = \sum_{x \in G} o(x),$$

defined by Amiri, Jafarian Amiri and Isaacs in [1]. Given a subgroup H of G , this has been generalized in [9] to the function

$$\psi_H(G) = \sum_{x \in G} o_H(x),$$

where $o_H(x)$ denotes the order of x relative to H , i.e., the smallest positive integer m such that $x^m \in H$. Clearly, for $H = 1$ we have $\psi_H(G) = \psi(G)$.

We remark that

$$\psi_H(G) = \sum_{x \in H} o_H(x) + \sum_{x \in G \setminus H} o_H(x) = |H| + \sum_{x \in G \setminus H} \frac{o(x)}{|\langle x \rangle \cap H|}$$

and therefore H is isolated in G if and only if

$$\psi_H(G) = |H| + \sum_{x \in G \setminus H} o(x) = |H| + \psi(G) - \psi(H).$$

Since for $H \triangleleft G$ we have $\psi_H(G) = |H|\psi(G/H)$, we infer that a normal subgroup H is isolated in G if and only if

$$(1.1) \quad \psi(G) - \psi(H) = |H|(\psi(G/H) - 1).$$

In particular, this equivalence holds for all subgroups H of a finite abelian group G . It will be used in what follows, together with Theorem 1 of [10]:

Theorem 1.2. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p -group, where $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$. Then*

$$(1.2) \quad \psi(G) = 1 + \sum_{\alpha=1}^{\alpha_k} (p^{2\alpha} f_{(\alpha_1, \alpha_2, \dots, \alpha_k)}(\alpha) - p^{2\alpha-1} f_{(\alpha_1, \alpha_2, \dots, \alpha_k)}(\alpha-1)),$$

where

$$f_{(\alpha_1, \alpha_2, \dots, \alpha_k)}(\alpha) = \begin{cases} p^{(k-1)\alpha} & \text{if } 0 \leq \alpha \leq \alpha_1, \\ p^{(k-2)\alpha + \alpha_1} & \text{if } \alpha_1 \leq \alpha \leq \alpha_2, \\ \vdots & \\ p^{\alpha_1 + \alpha_2 + \dots + \alpha_{k-1}} & \text{if } \alpha_{k-1} \leq \alpha. \end{cases}$$

We note that (1.2) gives a formula for the sum of element orders of an arbitrary finite abelian group because the function ψ is multiplicative. Also, we note that $\psi(G)$ in Theorem 1.2 is a polynomial in p of degree $2\alpha_k + \alpha_{k-1} + \dots + \alpha_1$. An alternative way of writing it is

$$(1.3) \quad \begin{aligned} \psi(G) &= p^{2\alpha_k + \alpha_{k-1} + \dots + \alpha_1} - (p-1) \sum_{\alpha=0}^{\alpha_k-1} p^{2\alpha} f_{(\alpha_1, \alpha_2, \dots, \alpha_k)}(\alpha) \\ &= p^{2\alpha_k + \alpha_{k-1} + \dots + \alpha_1} + \dots + p^{k+1} - p + 1. \end{aligned}$$

Most of our notation is standard and is usually not introduced here. Elementary notions and results on groups can be found in [4], [7], [8].

2. PROOFS OF THE MAIN RESULTS

We start with the following lemma whose proof is elementary and thus omitted.

Lemma 2.1. *Let G be a finite abelian group and H be a subgroup of G . Write G and H as the direct products of their Sylow subgroups*

$$G = G_1 \times G_2 \times \dots \times G_m \quad \text{and} \quad H = H_1 \times H_2 \times \dots \times H_m,$$

respectively. Then H is isolated in G if and only if there are $i_1, i_2, \dots, i_k \in \{1, 2, \dots, m\}$ such that H_{i_j} is isolated in G_{i_j} for all $j = 1, 2, \dots, k$ and $H_i = 1$ for all $i \neq i_1, i_2, \dots, i_k$.

Lemma 2.1 shows that our study can be reduced to finite abelian p -groups via the description of the structure of finite abelian groups.

Lemma 2.2. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p -group, where $1 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$, and H be a maximal subgroup of G . If H is isolated in G , then G is elementary abelian and H is a direct factor of G .*

Proof. Let $n = \alpha_1 + \alpha_2 + \dots + \alpha_k$. Since H is maximal and isolated in G , by the equality (1.1) it follows that

$$\psi(G) - \psi(H) = p^{n-1}(\psi(\mathbb{Z}_p) - 1) = p^{n+1} - p^n$$

and so $\psi(G)$ is a polynomial in p of degree $n+1$. On the other hand, by (1.3) we know that $\psi(G)$ is a polynomial in p of degree $2\alpha_k + \alpha_{k-1} + \dots + \alpha_1 = n + \alpha_k$. Thus $n + \alpha_k = n + 1$, that is $\alpha_k = 1$, implying that G is elementary abelian. The second conclusion is obvious. $\square$

From Lemma 2.2 we infer that if H is a proper isolated subgroup of a finite abelian p -group G , then

$$(2.1) \quad H \subset \Omega_1(G) = \{x \in G: x^p = 1\}$$

and, in particular, H is p -elementary abelian.

Indeed, take a subgroup K of G such that H is maximal in K . Then H is isolated in K and Lemma 2.2 shows that K must be elementary abelian, i.e., $K \subseteq \Omega_1(G)$. Hence, H is strictly contained in $\Omega_1(G)$.

Lemma 2.3. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p -group with $1 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_k$. Then G has no isolated proper subgroup.*

Proof. Assume that H is an isolated proper subgroup of G and let $|H| = p^m$. Then

$$(2.2) \quad \psi(G) - \psi(H) = p^m(\psi(G/H) - 1).$$

By (2.1) we know that H is elementary abelian and $m < k$. Then $\psi(H) = p^{m+1} - p + 1$ and therefore the left side of (2.2) becomes

$$\psi(G) - \psi(H) = p^{2\alpha_k + \alpha_{k-1} + \dots + \alpha_1} + \dots + p^{k+1} - p^{m+1}.$$

On the other hand, (2.1) shows that G/H has also k direct factors

$$G/H = \mathbb{Z}_{p^{\beta_1}} \times \mathbb{Z}_{p^{\beta_2}} \times \dots \times \mathbb{Z}_{p^{\beta_k}},$$

where either $\beta_i = \alpha_i$ or $\beta_i = \alpha_i - 1$ for all $i = 1, 2, \dots, k$. Thus, the right side of (2.2) becomes

$$\begin{aligned} p^m(\psi(G/H) - 1) &= p^m(p^{2\beta_k + \beta_{k-1} + \dots + \beta_1} + \dots + p^{k+1} - p) \\ &= p^{m+2\beta_k + \beta_{k-1} + \dots + \beta_1} + \dots + p^{m+k+1} - p^{m+1}. \end{aligned}$$

Hence, (2.2) leads to $p^{k+1} = p^{m+k+1}$, i.e., $m = 0$, a contradiction. $\square$

In the following, let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p -group, where $1 = \alpha_1 = \alpha_2 = \dots = \alpha_r < \alpha_{r+1} \leq \dots \leq \alpha_k$, and H be a subgroup of order p of G . Then H is isolated in G if and only if $\langle x \rangle \cap H = 1$ for all $x \in G \setminus H$, that is if and only if H is contained in no cyclic subgroup of order p^s with $s \geq 2$ of G . This is equivalent with the fact that H is not contained in $A = \mathbb{Z}_{p^{\alpha_{r+1}}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$.

The above remark can be easily extended to arbitrary proper subgroups H of G , namely H is isolated in G if and only if H contains no subgroup of order p of A .

Indeed, if H is not isolated in G , then there is $x \in G \setminus H$ such that $\langle x \rangle \cap H \neq 1$. Take a subgroup K of order p of $\langle x \rangle \cap H$. Then K is also not isolated in G and thus $K \subseteq A$. Consequently, H contains a subgroup of order p of A , a contradiction. The converse is obvious.

Hence, we proved the next lemma.

Lemma 2.4. *Let $G = \mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p -group, where $1 = \alpha_1 = \alpha_2 = \dots = \alpha_r < \alpha_{r+1} \leq \dots \leq \alpha_k$, and let $A = \mathbb{Z}_{p^{\alpha_{r+1}}} \times \dots \times \mathbb{Z}_{p^{\alpha_k}}$. Then a proper subgroup H of G is isolated in G if and only if $H \cap A = 1$.*

In particular, Lemma 2.4 shows that all subgroups of an elementary abelian p -group are isolated.

We are now able to prove our main result.

Proof of Theorem 1.1. It follows from Lemmas 2.3 and 2.4. □

Finally, we mention that the computation of isolated subgroups of finite abelian p -groups can be done by using well-known Goursat's lemma (see, e.g., the result (4.19) of [8]). We exemplify it in three particular cases:

Example 2.1.

- (1) The group $\mathbb{Z}_p \times \mathbb{Z}_{p^m}$ with $m \geq 2$ has $p+2$ isolated subgroups, namely 1, G and p subgroups of order p .
- (2) The group $\mathbb{Z}_p \times \mathbb{Z}_{p^m} \times \mathbb{Z}_{p^n}$ with $m, n \geq 2$ has p^2+2 isolated subgroups, namely 1, G and p^2 subgroups of order p .
- (3) The group $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^m}$ with $m \geq 2$ has $2p^2 + p + 2$ isolated subgroups, namely 1, G , $p^2 + p$ subgroups of order p and p^2 subgroups of order p^2 .

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